Generalized Kubo formula for spin transport: A theory of linear response to non-Abelian fields

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The traditional Kubo formula is generalized to describe the linear response with respect to non-Abelian fields. To fulfill the demand for studying spin transport, the SU(2) Kubo formula is derived by two conventional approaches with different gauge fixings. Those two approaches are shown to be equivalent where the nonconservation of the SU(2) current plays an essential role in guaranteeing the consistency. Some concrete examples relating spin Hall effect are considered. The dc spin conductivity in response to an SU(2) electric field vanishes in the system with parabolic unperturbed dispersion relation. By applying a time-dependent Rashba field, the spin conductivity can be measured directly. Our formula is also applied to the high-dimensional representation for the interests of some important models, such as Luttinger model and bilayer spin Hall system.

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I. INTRODUCTION

Kubo formula, one of the most important formulas in the linear response theory, has been widely used in condensed matter physics since it was derived by Kubo [1] for the electrical conductivity in solids. There are several kinds of Kubo formulas for the external fields to which the system responses are different. However, these formulas, such as those for the electrical conductivity and the susceptibility, all describe linear responses to the U(1) external fields.

Recently, a newly emerging field, spintronics [2, 3], has absorbed much attention for its promising applications in quantum information storage and processing. Spin Hall effect [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], as a candidate method to injecting spin current into semiconductors, is also discussed intensively. In this effect, the spin-orbit coupling is necessary no matter intrinsically or extrinsically. As this coupling can be regarded as a contribution of the SU(2) gauge potential [18], a new version of linear response theory in SU(2) formulation is necessary. Nevertheless, most of the previous works have mainly focused on the linear response of such system to an external electric field, and hence the traditional Kubo formula was adopted directly except Ref [19] which dealt with non-Abelian response and considered spin Hall effect in the presence of an SU(2) gauge field. There were some papers [20, 21] discussing the responses to a spin-orbit coupling with spatially varying strength, but the authors employed other approaches rather than Kubo formula as the SU(2) Kubo formula has not been established. It thus becomes inevitable to develop a generalized Kubo formula so that the linear response to the external SU(2) gauge fields can be evaluated.

In present paper, we derive a formula which describes the linear response to an SU(2) external field using the strategy ever employed by Kubo for the U(1) case. It is not a straightforward derivation since the algebra is totally different. Especially, the expression of the SU(2) "electric field " evolves one more term of gauge potentials [22] than the U(1) case due to its non-Abelian feature. It seems obscure to directly find the equivalence between the Kubo formulas derived with different gauge fixings. We will show that the extra term in the SU(2) "electric field " precisely corresponds to the nonvanishing term in the "continuity-like" equation [18] which includes the spin procession [12, 18]. Its origin stems from the definition of the conserved current [18] in the presence of the SU(2) field. Since one of recent research interests focuses on the spintronics, some explicit examples in spin Hall systems are discussed in terms of our SU(2) Kubo formula, such as the spin susceptibility and current in response to the effective spin-orbit coupling [13, 14]. In spin Hall effect, the spin conductivity is believed to be canceled by the effect of disorder in two-dimensional electron gas [8]. It is due to the parabolic unperturbed dispersion relation [25]. In such a system, the spin current in response to the external spin-orbit coupling also vanishes. Then the systems with nonparabolic unperturbed dispersion relation become significant. In such systems, the spin conductivity in response to either U(1) or SU(2) external fields does not vanish. An experimentally accessible case is also given in which the spin conductivity is related to the dielectric function. We also extend the application of our formula to a high-dimensional representation, saying spin-3/2 representation, which is related to some important systems, such as the Luttinger model [23] and the bilayer systems [24]. The spin conductivity in the Luttinger model vanishes, which describes the response to the effective field of structural inversion asymmetry.

The paper is organized as follows. In Sec. II, we derive a general Kubo formula with respect to a single-frequency SU(2) external field at zero temperature. Then we show in Sec. III that this formulation is consistent with the one by choosing a zero-frequency external field at the very beginning. In Sec. IV, we give the applications of our SU(2) Kubo formula to some models of spin-1/2 representation. In Sec. V, our theory is applied to a high-

dimensional representation (i.e., spin-3/2 representation) system. Several concrete example models are also given. In Sec. VI, we give a brief summary with some remarks. In the appendixes, we give the detailed calculations of the correlation functions in Matsubara formalism.

II. SU(2) KUBO FORMULA AT ZERO TEMPERATURE

A generalized total Hamiltonian for a variety of models to study spin transport can be written as [18]

$$H = \frac{1}{2m}\sum_l \Big(\mathbf{p}_l - \frac{e}{c}\mathbf{A}(\mathbf{r}_l,t) - \eta \boldsymbol{\mathcal{A}}^a(\mathbf{r}_l,t)\tau^a\Big)^2 + \hat{V}_{dis}, \quad (1)$$

where $\hat{V}_{dis}$ is the potential caused by disorders. Throughout this paper, the index l refers to the lth particle, a, b, and c refer to spin space while i and j the spatial space, and repeated indices are summed over. τ^a stands for the generators of SU(2) algebra, $\mathbf{A}$ and $\boldsymbol{\mathcal{A}}^a$ are the U(1) and SU(2) gauge potentials, respectively. Usually these gauge potentials consist of two parts, internal and external fields. In order to derive a general formula for the conductivity in response to an SU(2) external "electric field ", we separate the Hamiltonian (1) into two parts, $H = H_0 + H'$, with

$$\begin{aligned} H_0 &= \frac{1}{2m}\sum_l \hat{\boldsymbol{\pi}}_l^2 + \hat{V}_{dis}, \\ H' &= \frac{-\eta}{2m}\sum_l \Big(\hat{\boldsymbol{\pi}}_l \cdot \boldsymbol{\mathcal{A}}^a_{\text{ext}}(\mathbf{r}_l,t)\tau^a + \tau^a \boldsymbol{\mathcal{A}}^a_{\text{ext}}(\mathbf{r}_l,t)\cdot\hat{\boldsymbol{\pi}}_l\Big), \end{aligned} \quad (2)$$

where the operator $\hat{\boldsymbol{\pi}}_l = \mathbf{p}_l - \frac{e}{c}\mathbf{A}_{\text{int}}(\mathbf{r}_l,t) - \eta\boldsymbol{\mathcal{A}}^a_{\text{int}}(\mathbf{r}_l,t)\tau^a$ stands for the dynamical momentum involving internal U(1) and SU(2) potentials if any. Note that this separation is up to the first order of $\boldsymbol{\mathcal{A}}^a_{\text{ext}}(\mathbf{r}_l,t)$.

The relation between SU(2) "electric fields " and the gauge potentials is given by

$$E^a_i = -\partial_0 \mathcal{A}^a_i - \partial_i \mathcal{A}^a_0 + \eta\epsilon^{abc}\mathcal{A}^b_0\mathcal{A}^c_i, \quad (3)$$

where η is the coupling constant. The SU(2) "electric field " [18] is expected to be realized by spatially [21] or timely [20] dependent Rashba or Dresselhaus coupling strength. We consider the linear response with respect to the "electric" components of a non-Abelian field, $\mathbb{E}_i = E^a_i\tau^a$. For simplicity, we take the SU(2) external field to be of single-frequency, namely,

$$E^a_i(\mathbf{r},t) = E^a_i(\mathbf{q},\omega)\ e^{i\mathbf{q}\cdot\mathbf{r}-i\omega t}. \quad (4)$$

The problem involving the external field of an arbitrary form on $\mathbf{r}$ only differs from this case by a fourier transform. As is well-known, the external field of frequency$(\mathbf{q},\omega)$ actually reads $E^a_i(\mathbf{q},\omega)\cos(\mathbf{q}\cdot\mathbf{r}-\omega t)$ which is real valued. The expression in Eq. (4) is convenient for calculation. Thus in the subsequent formulas, only the real parts have the physical meaning.

Now we choose a gauge that $\mathcal{A}^a_0 = 0$, which corresponds to the temporal gauge in the U(1) case, then the SU(2) external "electric field" comes into the Hamiltonian through the vector potential $\mathcal{A}^a_i$,

$$\mathcal{A}^a_i(\mathbf{r},t) = \frac{1}{i\omega}E^a_i(\mathbf{r},t) = \frac{1}{i\omega}E^a_i(\mathbf{q},\omega)e^{i\mathbf{q}\cdot\mathbf{r}-i\omega t}. \quad (5)$$

Hereafter, we omit the subscript specifying the external field for simplicity.

Based on the definition for single particle, one can define the SU(2)-current operator for the unperturbed system

$$\hat{\mathbf{J}}^a(\mathbf{r}) = \frac{\eta}{4m}\sum_l \Big[\{\tau^a,\hat{\boldsymbol{\pi}}_l\}\delta(\mathbf{r}-\mathbf{r}_l) + \delta(\mathbf{r}-\mathbf{r}_l)\{\tau^a,\hat{\boldsymbol{\pi}}_l\}\Big], \quad (6)$$

where the curl bracket denotes anticommutator and the velocity operator of the lth particle determined by the Heisenberg equation of motion, $\hat{\boldsymbol{\pi}}_l/m = [\mathbf{r}_l,\ H_0]/(i\hbar)$ is matrix-valued for the SU(2) case. In terms of this current, the perturbation Hamiltonian H' can be expressed as

$$\begin{aligned} H' &= -\int d\mathbf{r}\ \hat{J}^a_i(\mathbf{r})\mathcal{A}^a_i(\mathbf{r},t) \\ &= -\frac{1}{i\omega}\ \hat{J}^a_i(\mathbf{q})\ E^a_i(\mathbf{q},\omega)e^{-i\omega t}, \end{aligned} \quad (7)$$

where $\hat{J}^a_i(\mathbf{q})$ is the Fourier image of $\hat{J}^a_i(\mathbf{r})$. Clearly, the interaction term is the product of the SU(2) current and the SU(2) "electric field ".

Taking the perturbation of the external field into account, we have $\mathbf{\Pi}_l = \hat{\boldsymbol{\pi}}_l - \eta\boldsymbol{\mathcal{A}}^a(\mathbf{r}_l)\tau^a$. Then the total SU(2) current driven by the external SU(2) "electric field " reads

$$\begin{aligned} \boldsymbol{\mathcal{J}}^a(\mathbf{r},t) &= \frac{\eta}{4m}\sum_l\Big[\{\tau^a,\mathbf{\Pi}_l\}\delta(\mathbf{r}-\mathbf{r}_l) + \delta(\mathbf{r}-\mathbf{r}_l)\{\tau^a,\mathbf{\Pi}_l\}\Big] \\ &= \hat{\mathbf{J}}^a(\mathbf{r}) - \frac{\eta^2}{4m}\boldsymbol{\mathcal{A}}^a(\mathbf{r},t)\ n_0, \end{aligned} \quad (8)$$

where n_0 is the particle density.

At zero temperature, this SU(2) current is evaluated for the ground state of the system. In the interaction representation, the state $|\psi(t)\rangle$ of the system at time t is related to the eigenvector $|\phi\rangle$ of H_0 by the S-matrix, i.e., $|\psi(t)\rangle = S(t,-\infty)|\phi\rangle$. Up to the linear order of $H'_I(t')$, $S(t,-\infty) = 1 - \frac{i}{\hbar}\int_{-\infty}^t dt' H'_I(t')$, where $H'_I(t') = e^{iH_0t'/\hbar}H'e^{-iH_0t'/\hbar}$. Then the average of the first term in the total SU(2) current Eq. (8) is given by

$$\begin{aligned} &\langle \hat{J}^a_i(\mathbf{r},t)\rangle = \langle \hat{J}^a_i(\mathbf{r},t)\rangle_0 - \frac{i}{\hbar}\int_{-\infty}^t dt' \langle[\hat{J}^a_i(\mathbf{r},t), H'_I(t')\ ]\rangle_0 \\ &= \frac{E^b_j(\mathbf{r},t)}{\hbar\omega}\int_{-\infty}^t dt' e^{i\omega(t-t')}e^{-i\mathbf{q}\cdot\mathbf{r}}\langle[\ \hat{J}^a_i(\mathbf{r},t), \hat{J}^b_j(\mathbf{q},t')]\rangle_0, \end{aligned} \quad (9)$$

where $\langle \hat{J}_i^a(\mathbf{r},t)\rangle_0$ has been dropped since no SU(2) current is considered to follow in the absence of the external fields. Together with the second term, we obtain the following expression:

$$\begin{aligned}\langle \mathcal{J}_i^a(\mathbf{r},t)\rangle &= \langle \hat{J}_i^a(\mathbf{r},t)\rangle - \frac{\eta^2}{4m}\mathcal{A}_i^a(\mathbf{r},t)\ n_0 \\ &= \frac{E_j^b(\mathbf{r},t)}{\hbar\omega}\int_{-\infty}^{t} dt' e^{i\omega(t-t')}e^{-i\mathbf{q}\cdot\mathbf{r}}\langle[\hat{J}_i^a(\mathbf{r},t),\hat{J}_j^b(\mathbf{q},t')]\rangle_0 \\ &\quad + \frac{i\eta^2 n_0}{4m\omega}\delta^{ab}\delta_{ij}E_j^b(\mathbf{r},t) \\ &\equiv \sigma_{ij}^{ab}(\mathbf{q},\omega;\mathbf{r})E_j^b(\mathbf{r},t). \end{aligned} \tag{10}$$

Since the conductivity represents the property of the whole system, we need take the average over the system to get the SU(2) conductivity,

$$\begin{aligned}\sigma_{ij}^{ab}(\mathbf{q},\omega) &= \frac{1}{\hbar\omega V}\int_{-\infty}^{t} dt'\ e^{i\omega(t-t')}\langle[\hat{J}_i^{\dagger a}(\mathbf{q},t),\hat{J}_j^b(\mathbf{q},t')]\rangle_0 \\ &\quad + \frac{i\eta^2\ n_0}{4m\omega}\delta^{ab}\delta_{ij}, \end{aligned} \tag{11}$$

with V the volume of the system. The spin conductivity here is a tensor in spin space rather than a vector as in the case of linear response to the U(1) external field.

As a conventional strategy, a retarded current-current correlation function is thus introduced to calculate this conductivity,

$$Q_{ij}^{ab}(\mathbf{q},t-t') = -\frac{i}{V}\ \theta(t-t')\langle[\ \hat{J}_i^{\dagger a}(\mathbf{q},t),\ \hat{J}_j^b(\mathbf{q},t')\ ]\rangle_0, \tag{12}$$

where $\theta(t-t')$ is the step function which vanishes unless $t>t'$. The Fourier transform of Eq. (12) is given by

$$\begin{aligned}&Q_{ij}^{ab}(\mathbf{q},\omega) \\ &= -\frac{i}{V}\int_{-\infty}^{+\infty} dt\ \theta(t-t')e^{i\omega(t-t')}\langle[\hat{J}_i^{\dagger a}(\mathbf{q},t),\ \hat{J}_j^b(\mathbf{q},t')\ ]\rangle_0. \end{aligned} \tag{13}$$

Comparing with Eq. (11), we obtain

$$\sigma_{ij}^{ab}(\mathbf{q},\omega) = \frac{i}{\hbar\omega}\Big[Q_{ij}^{ab}(\mathbf{q},\omega) + \frac{\hbar\eta^2 n_0}{4m}\delta^{ab}\delta_{ij}\Big]. \tag{14}$$

To simplify the calculations, we introduce a Matsubara function $Q_{ij}^{ab}(\mathbf{q},i\nu)$ which reduces to the retarded correlation function $Q_{ij}^{ab}(\mathbf{q},\omega)$ by changing $i\nu$ to $\omega+i\delta$,

$$Q_{ij}^{ab}(\mathbf{q},i\nu) = -\frac{1}{V}\int_0^{\beta} du\ e^{i\nu u}\langle T_u\ \hat{J}_i^{\dagger a}(\mathbf{q},u)\hat{J}_j^b(\mathbf{q},0)\rangle, \tag{15}$$

where T_u denotes the u-ordering operator and $\beta = (k_BT)^{-1}$ with k_B the Boltzmann constant. We thus have derived a generalized Kubo formula for spin transport in response to an external SU(2) "electric field ".

III. AN EQUIVALENT FORMULATION FOR ZERO FREQUENCY

In the previous section, we derived the SU(2) Kubo formula choosing the gauge potential $\mathcal{A}_0^a = 0$. To obtain the dc conductivity, one just needs to take the limit $\omega \to 0$. As is well-known in the conventional electrical conductivity, the Kubo formula can also be derived alternately by choosing a constant external field as a start point. The continuity equation for electric charge conservation guarantees the two derivations to be equivalent. Whereas, in the SU(2) case, the current defined by Eq. (6) is not conserved as long as an SU(2) interaction is present. For example, the spin current, a special SU(2) current with $\eta=\hbar$, is not conserved if there exists the Zeeman term or spin-orbit coupling. In this case, the continuity equation does not hold [18, 19], instead, we have the following relation:

$$\Big(\frac{\partial}{\partial t} - \eta\vec{\mathcal{A}}_0\times\Big)\vec{\sigma}(\mathbf{r},\ t) + \Big(\frac{\partial}{\partial x_i} + \eta\vec{\mathcal{A}}_i\times\Big)\vec{J}_i(\mathbf{r},\ t) = 0, \tag{16}$$

where $\sigma^a(\mathbf{r}) = \eta\psi^\dagger(\mathbf{r})\tau^a\psi(\mathbf{r})$ and $\mathbf{J}^a(\mathbf{r},t)$ are the SU(2) density and current respectively, and notations $\vec{\sigma} = (\sigma^1,\ \sigma^2,\ \sigma^3)$, $\vec{\mathcal{A}} = (\mathcal{A}^1,\ \mathcal{A}^2,\ \mathcal{A}^3)$ etc. are adopted. Unlike the charge current which is conserved, the spin current is not conserved, thus a natural question is whether the SU(2) Kubo formula we derived in the previous section is still consistent with the other derivation?

Now let us choose $\partial_0\mathcal{A}_i^a = 0$ for the zero frequency case, then the SU(2) electric field and the perturbation Hamiltonian are given by

$$E_i^a = -\partial_i\mathcal{A}_0^a + \eta\epsilon^{abc}\mathcal{A}_0^b\mathcal{A}_i^c, \tag{17}$$

and

$$H' = \int d\mathbf{r}\ \sigma^a(\mathbf{r},t)\mathcal{A}_0^a(\mathbf{r}). \tag{18}$$

By means of the method suggested by Luttinger, the total SU(2) current can be obtained once the density matrix ρ is introduced. The density matrix including the deviations caused by the perturbation takes the form

$$\rho(t) = \rho_0 + \delta\rho(t), \tag{19}$$

where ρ_0 refers to the density matrix with respect to the unperturbed Hamiltonian and $\delta\rho(t)$ is brought about by the perturbation one, H'. From the equation of motion for the perturbed part of the density matrix,

$$i\hbar\frac{\partial\delta\rho(t)}{\partial t} = [H_0,\delta\rho(t)] + [H',\ \rho_0], \tag{20}$$

we can obtain a solution for $\delta\rho(t)$

$$\delta\rho(t) = -\frac{1}{\hbar}\int_0^{\infty} dt\int_0^{\beta} d\beta'\ \rho_0\frac{\partial}{\partial t}H_I'(-t-i\beta'). \tag{21}$$

With the help of the density matrix, the SU(2) current can be then evaluated by taking the average

$$\langle \hat{J}_i^a(\mathbf{r},t)\rangle = \mathrm{tr}\big(\rho(t)\hat{J}_i^a(\mathbf{r})\big)$$
$$= -\frac{1}{\hbar}\int_0^\infty dt\int_0^\beta d\beta' \mathrm{tr}\Big[\rho_0\,\frac{\partial}{\partial t}H'_I(-t-i\beta')\;\hat{J}_i^a(\mathbf{r})\Big], \tag{22}$$

where the equilibrium part of the current $\mathrm{tr}(\rho_0\hat{J}_i^a(\mathbf{r}))$ is assumed to be zero. The derivative of H'_I with respect to time t is calculated as

$$\partial_t H'_I(-t) = \int d\mathbf{r}\;\partial_t\sigma^a(\mathbf{r},-t)\mathcal{A}_0^a(\mathbf{r}). \tag{23}$$

Using the "continuity-like" equation (16) and integration by parts, we have

$$\begin{aligned}\partial_t H'_I(-t) \;&=\; \int d\mathbf{r}\Big(\eta\epsilon^{abc}(\mathcal{A}_0^b\,\sigma^c - \mathcal{A}_i^b J_i^c) - \partial_i J_i^a\Big)\mathcal{A}_0^a \\ &=\; -\int d\mathbf{r}\; E_i^a J_i^a(\mathbf{r},-t),\end{aligned} \tag{24}$$

where we did not write out the arguments in the first line for simplicity. Substituting it into Eq. (22), we obtain

$$\langle \hat{J}_i^a(\mathbf{r},t)\rangle$$
$$=\frac{1}{\hbar}\int_0^\infty dt\int_0^\beta d\beta'\int d\mathbf{r}'\;\mathrm{tr}\Big[\rho_0 E_j^b J_j^b(\mathbf{r}',-t-i\beta')\hat{J}_i^a(\mathbf{r})\Big]. \tag{25}$$

Consequently, the dc SU(2) conductivity is obtained from the above equation after integrating $\mathbf{r}$ over the volume V,

$$\sigma_{ij}^{ab} = \frac{1}{\hbar V}\int_0^\infty dt\int_0^\beta d\beta'\;\mathrm{tr}\Big[\rho_0\;J_j^b(-t-i\beta')\;\hat{J}_i^a\Big]. \tag{26}$$

This result is obviously independent on the frequency. It is also consistent with the one which we derived in the previous section once we introduce the representation of the eigenstates $|n\rangle$ of H_0. Note that the spin procession terms, $\eta\vec{\mathcal{A}}_i\times\vec{J}_i(\mathbf{r},\;t)-\eta\vec{\mathcal{A}}_0\times\vec{\sigma}(\mathbf{r},\;t)$, precisely compensate the second term of Eq. (17), which makes our theory self-consistent. Since the SU(2) "electric field " includes an extra term of gauge potential in comparison to the U(1) field, the nonconservation of the SU(2) current plays an essential role in guaranteeing the consistency. That is to say, the SU(2) current exactly responds to the SU(2) "electric field " no matter which gauge is chosen.

IV. APPLICATIONS FOR SPIN-1/2 REPRESENTATION

From now on, we will give some applications of our SU(2) Kubo formula. In this section, we mainly focus on the examples in spin-1/2 representation without impurities and in the limit $q\to 0$.

A. Spin susceptibility

Spin is a category of SU(2) entity with $\eta=\hbar$. The spin degree of freedom is discussed extensively in recent years for its promising application. The effective spin-orbit coupling emerged significantly in some semiconductors [13, 14] is of importance for its possible manipulating of spin. Using our SU(2) Kubo formula, we can directly calculate the spin susceptibility which describes the linear response of the spin density to the spin-orbit coupling.

The spin susceptibility χ_i^{ab} is defined as

$$\langle\hat{S}^a\rangle = \chi_i^{ab}E_i^b, \tag{27}$$

where $\hat{S}^a = \hbar\sum_k C_k^\dagger\tau^a C_k$ is the spin density. Here we adopted a simplified notion $C_k^\dagger = (C_{k\uparrow}^\dagger,\;C_{k\downarrow}^\dagger)$ with $C_{k\uparrow}^\dagger$ creating a spin-up particle of momentum k etc. The corresponding retarded correlation function in Matsubara formalism $\Pi_i^{ab}(i\nu)$ is given by

$$\Pi_i^{ab}(i\nu) = -\frac{1}{V}\int_0^\beta du\;e^{i\nu u}\langle \mathrm{T_u\,\hat{S}^a(u)\hat{J}_i^b(0)}\rangle. \tag{28}$$

Hereafter, we take the unperturbed Hamiltonian to be $H_0=\sum_k C_k^\dagger(\varepsilon(k)+d^a(k)\tau^a)C_k$ for its elegant form in Green's function. The second term represents the internal SU(2) field with d^a the components of this field. This system has two bands, $E_-=\varepsilon(k)+|d|$ and $E_+=\varepsilon(k)-|d|$, with $|d|=\sqrt{d^a\,d^a}$. In the limit $\omega\to 0$, the susceptibility reads

$$\chi_i^{ab} = \frac{\hbar}{2V}\sum_k\frac{n_{F_-}-n_{F_+}}{|d|^3}\;\epsilon^{abc}d^c\frac{\partial\varepsilon(k)}{\partial k_i}, \tag{29}$$

where n_{F_-} and n_{F_+} are the Fermi distribution functions and "$-,+$" label the different bands. This result is antisymmetric to the indices labeling spin degree of freedom. Using this result, we calculate the spin susceptibilities with two kinds of internal fields, Rashba and Dresselhaus couplings,

$$\begin{aligned}H_{\scriptscriptstyle R} \;&=\; -2\alpha(k_x\tau^y - k_y\tau^x),\\ H_{\scriptscriptstyle D} \;&=\; -2\beta(k_x\tau^x - k_y\tau^y).\end{aligned} \tag{30}$$

These two kinds of couplings dominate in narrow gap semiconductors such as GaAs and here we take their two-dimensional (2D) forms to represent the effective spin-orbit couplings in two-dimensional electron gas (2DEG). In these cases, the components χ_i^{xy} vanish since $d^z=0$. The results are shown in Table I, where we have taken the usual parabolic form that $\varepsilon(k)=\hbar^2k^2/2m$.

B. Spin conductivity

With $\eta=\hbar$, the spin current reads

$$\hat{J}_i^a = \frac{1}{2}\sum_k C_k^\dagger\Big\{\frac{\partial\varepsilon(k)}{\partial k_i}+\frac{\partial d^b}{\partial k_i}\tau^b,\tau^a\Big\}C_k. \tag{31}$$

SU(2) internal field	χ_x^{zx}	χ_x^{zy}	χ_y^{zx}	χ_y^{zy}
Rashba ($\frac{\hbar}{32\pi\alpha}$)	1	0	0	1
Dresselhaus ($\frac{\hbar}{32\pi\beta}$)	0	-1	-1	0

TABLE I: Spin susceptibilities: α and β are coupling constants for the Rashba and Dresselhaus coupling respectively.

SU(2) internal field	σ_{xx}^{zx}	σ_{xx}^{zy}	σ_{xy}^{zx}	σ_{xy}^{zy}	σ_{yy}^{zx}	σ_{yy}^{zy}
Rashba ($\frac{\hbar^2}{16\pi m\alpha}$)	c	0	$\frac{c'}{2}$	$\frac{c}{2}$	0	c'
Dresselhaus ($\frac{\hbar^2}{16\pi m\beta}$)	0	-c	$-\frac{c}{2}$	$-\frac{c'}{2}$	$-c'$	0

TABLE II: Spin conductivities: c and c' represent the displacements in k space.

After calculating the Matsubara function (see Appendix A) and changing $i\nu \to \omega$, we derive the conductivity

$$\sigma_{ij}^{ab} = \frac{1}{2V}\sum_k \frac{n_{F-} - n_{F+}}{|d|^3}\epsilon^{abc}d^c \frac{\partial\varepsilon(k)}{\partial k_i}\frac{\partial\varepsilon(k)}{\partial k_j}. \qquad (32)$$

This expression manifests that the conductivity is antisymmetric to the spin indices a, b and symmetric with respect to the spatial indices i, j with the parabolic dispersion relation. Note that when the U(1) part of H_0 is parabolic, i.e., $\varepsilon(k) = \hbar^2k^2/2m$, and d^a is linear of k^a, the summation over k vanishes. Since the conventional spin-orbit couplings are Rashba and Dresselhaus couplings, which contain no quadratic terms of k, we should consider $\varepsilon(k) = \frac{\hbar^2}{2m}\left[(k_x+c)^2+(k_y+c')^2\right]$ for nonvanishing results, which represents a shift of momentum k in the material. Table II shows the spin conductivities with two kinds of internal fields.

At this stage, it is worthwhile to recall some previous work in spin Hall effect. Up to now, a general consensus is made that the spin conductivity in response to an external Maxwell electric field is exactly canceled by the effect of disorder in two-dimensional electron gas with spin-orbit coupling. The cancellation is due to the parabolic form of unperturbed band structure [25]. It is worthwhile to point out that our SU(2) conductivity also vanishes when $\varepsilon(k)$ takes the parabolic form even in the absence of disorder. It is an essential difference that our conductivity refers to the linear response to an external Yang-Mills electric field which is also a vector in SU(2) Lie algebra space whose bases, the Pauli matrices, are anticommute. Anyway the system with nonparabolic dispersion relation is of great importance, since the conductivity, no matter in the usual spin Hall effect with disorder or derived by our SU(2) Kubo formula without disorder, is expected to be observed in experiments.

Finally, we will consider an experimentally available case. Since the Rashba coupling strength can be tuned by the gate voltage applied to 2DEG, we take a Rashba coupling with time-dependent strength as the external SU(2) field and Dresselhaus coupling as an internal field. Then we can get an ac conductivity depending on the frequency ω. The result reads

$$\sigma_{xx}^{zy}(\omega) = -\frac{c\hbar^6\omega^2}{32m^2\beta^3\pi}\epsilon_D(\omega), \qquad (33)$$

where $\epsilon_D(\omega)$ is the dielectric function caused by the Dresselhaus spin-orbit coupling [26], namely,

$$\epsilon_D(\omega) = \frac{4\beta^3}{\hbar^2\omega^2}\int_{k_{F+}}^{k_{F-}} \frac{k^2dk}{(2\beta k)^2 - \hbar^2\omega^2}. \qquad (34)$$

This dielectric function is a macroscopic quantity and can be directly measured. Carrying out the integration over k gives a resonant result,

$$\sigma_{xx}^{zy} = -\frac{c\hbar^2}{16\pi\beta m} - \frac{c\hbar^4\omega}{128\pi\beta^3m^2}\ln\left|\frac{k-\hbar\omega/2\beta}{k+\hbar\omega/2\beta}\right|_{k_{F+}}^{k_{F-}}, \qquad (35)$$

where k_F- and k_F+ refer to the Fermi momenta of both bands. The same resonance is also shown in Ref. [27]. Other components are given by $\sigma_{xy}^{zx} = \frac{1}{2}\sigma_{xy}^{zy}$ while σ_{yy}^{zx} and σ_{xy}^{zy} differ from them by $c \to c'$.

V. APPLICATIONS FOR SPIN-3/2 REPRESENTATION

In the previous section, we have discussed several examples using the SU(2) Kubo formula in spin-1/2 representation. It is well- known that there exit many important systems which carry out the spin-3/2 representation of the SU(2) algebra, for example, the Luttinger model [23] containing the intrinsic spin-orbit coupling, bilayer systems [24] taking into account of spin degree of freedom, etc. Thus it is worthwhile for us to extend our discussion to high-dimensional representations, such as spin-3/2 representation. The examples mentioned above are also discussed, which may be instructive for the experiments.

A. General consideration

The spin-3/2 representation of SU(2) generators read

$$\tau^x = \begin{pmatrix} 0 & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{pmatrix},$$

$$\tau^y = \begin{pmatrix} 0 & -i\sqrt{3}/2 & 0 & 0 \\ i\sqrt{3}/2 & 0 & -i & 0 \\ 0 & i & 0 & -i\sqrt{3}/2 \\ 0 & 0 & i\sqrt{3}/2 & 0 \end{pmatrix},$$

$$\tau^z = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}. \qquad (36)$$

To simplify the calculations, we use a convenient representation of the Clifford algebra adopted in Ref. [28], namely, $\Gamma^1 = \sigma^z \otimes \sigma^y$, $\Gamma^2 = \sigma^z \otimes \sigma^x$, $\Gamma^3 = \sigma^y \otimes I$, $\Gamma^4 = \sigma^x \otimes I$ and $\Gamma^5 = \sigma^z \otimes \sigma^z$ where the σ's are the 2 by 2 Pauli matrices. These gamma matrices satisfy $\{\Gamma^\alpha, \Gamma^\beta\} = 2\delta_{\alpha\beta}$ and $\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5 = -1$. Hereafter, the Greek indices α, β, γ, etc. run from 1 to 5. This representation can be obtained from the Dirac representation of the gamma matrices by a unitary transformation. Those five matrices can compose ten antisymmetric matrices $\Gamma^{\alpha\beta} = [\Gamma^\alpha, \Gamma^\beta]/2i$ which constitute the spinor representation of SO(5) algebra. The spin-3/2 operators can be expressed as a linear combination of those $\Gamma^{\alpha\beta}$, i.e., $\tau^a = \frac{1}{4i} L^a_{\alpha\beta}\Gamma^{\alpha\beta}$, where the coefficients $L^a_{\alpha\beta} = -L^a_{\alpha\beta}$ are just the five-dimensional representation of the SU(2) algebra (i.e., spin-2 representation of the angular momentum operators).

Since Γ^α, $\Gamma^{\alpha\beta}$ together with the identity I span the space of 4×4 Hermitian matrices, one can write out a general Hamiltonian in spin-3/2 representation in terms of those gamma matrices,

$$H_0 = \sum_k C_k^\dagger(\varepsilon(k) + d^\alpha(k)\Gamma^\alpha)C_k, \tag{37}$$

where $C_k^\dagger = (C_{k,1}^\dagger, C_{k,2}^\dagger, C_{k,3}^\dagger, C_{k,4}^\dagger)$ with the second index referring to either spin-band or spin-layer labels. Here we do not include the linear combination of $\Gamma^{\alpha\beta}$ which makes the Green's functions difficult to calculate. For this unperturbated Hamiltonian, there exist two types of perturbation part H'. One is constructed by $\Gamma^{\alpha\beta}$ and the other by Γ^α. The problem relating the spin current in Luttinger model in response to the effective field caused by the structure inversion asymmetry is of the first type. In this case, the structure inversion asymmetry is taken as the perturbation and hence

$$H' = \sum_k C_k^\dagger h^a \tau^a C_k. \tag{38}$$

Then the linear response of the spin current to H' reads

$$\langle \hat{J}_i^a \rangle = \sigma_i^{ab} h^b. \tag{39}$$

In calculating the retarded correlation function $Q_i^{ab}(i\nu)$, we will encounter

$$\begin{aligned}&\mathrm{tr}\Big(G(k,i\omega_n)\hat{J}_i^a G(k,i\omega_n+i\nu)\tau^b\Big)\\ &= -\frac{1}{16}\mathrm{tr}\Big(G(k,i\omega_n)\Big\{\frac{\partial\varepsilon(k)}{\partial k_i} + \frac{\partial d^\alpha}{\partial k_i}\Gamma^\alpha,\ L^a_{\beta\gamma}\Gamma^{\beta\gamma}\Big\}G(k,i\omega_n+i\nu)\ L^b_{\mu\nu}\Gamma^{\mu\nu}\Big),\end{aligned} \tag{40}$$

where $G(k, i\omega_n)$ is the Matsubara function for which the detailed calculation is given in Appendix B.

Since the traces of gamma matrices are always real, the appearance of double $\Gamma^{\alpha\beta}$ matrices makes Eq. (40) real and the summation of the Matsubara function also gives no imaginary contribution after changing $i\nu \to \omega$. This directly results in a vanishing spin conductivity.

The second type of perturbation is constructed by Γ^α,

$$H' = \sum_k C_k^\dagger h^\alpha \Gamma^\alpha C_k, \tag{41}$$

and later we will discuss some concrete examples. The SU(2) Kubo formula is then

$$\langle J_i^b \rangle = \sigma_i^{b\alpha} h^\alpha. \tag{42}$$

The corresponding retarded correlation function is given by

$$Q_i^{b\alpha}(i\nu) = -\frac{1}{V}\int_0^\beta du\ e^{i\nu u}\langle \mathrm{T_u}\ \hat{\mathrm{J}}_\mathrm{i}^{\dagger\mathrm{b}}(\mathrm{u})\hat{\Gamma}^\alpha(0)\rangle_0. \tag{43}$$

After changing $i\nu \to \omega$ and taking the limit $\omega \to 0$, we obtain the dc conductivity

$$\sigma_i^{b\alpha} = \frac{-\eta}{4\hbar V}\sum_k \mathrm{Im}\Big(2\frac{\partial\varepsilon(k)}{\partial k_i}L^b_{\alpha\beta}d^\beta - \epsilon^{\alpha\beta\gamma\mu\nu}L^b_{\beta\gamma}\frac{\partial d^\mu}{\partial k_i}d^\nu\Big)\frac{n_{F_-} - n_{F_+}}{|d|^3}. \tag{44}$$

B. Concrete examples

Now we are in the position to discuss two concrete examples with the second type of H'. First, we calculate the spin conductivity for a bilayer system undergoing the

Rashba coupling along opposite directions,

$$H_0 = \varepsilon(k) + \alpha\sigma^z \otimes (k_x\sigma^y - k_y\sigma^x) + \xi\sigma^x \otimes I. \qquad (45)$$

In this model the tunneling between the two layers is included in which ξ accounts for the tunneling strength. We take the SU(2) flux to be the external perturbation, $H' = \phi\ \sigma^z \otimes \sigma^z$, i.e., $h_5 = \phi$. A direct calculation of the spin conductivity gives the following result:

$$\begin{aligned}
&\sigma_i^{z5} = 0, \\
&\sigma_x^{x5} = -\frac{1}{4\pi\alpha\sqrt{\alpha^2k^2+\xi^2}}\Big(\frac{\sqrt{3}}{4}\frac{\hbar^2(\alpha^2k^2+2\xi^2)}{m\alpha^2} + \xi\Big)_{k_{F+}}^{k_{F-}}, \\
&\sigma_y^{y5} = -\sigma_x^{x5}.
\end{aligned} \qquad (46)$$

In the limit $\xi \to 0$, σ_x^{x5} reduces to $-\frac{\sqrt{3}}{4\pi\alpha}$.

As another example, we take the tunnelling term to be the perturbation, that is $H' = \xi'\sigma^x \otimes I$ and $H_0 = \varepsilon(k) + \alpha\sigma^z \otimes (k_x\sigma^y - k_y\sigma^x)$, correspondingly, $h^4 = \xi'$. Then we have the following result:

$$\begin{aligned}
\sigma_\alpha^{z4} &= 0, \\
\sigma_x^{x4} &= -\frac{1}{8\pi\alpha}, \\
\sigma_y^{y4} &= -\sigma_x^{x4}.
\end{aligned} \qquad (47)$$

VI. SUMMARY AND REMARKS

In this paper, we have generalized the Kubo formula to describe the linear response of the SU(2) current to the external SU(2) "electric field", which traditionally describes the one to the U(1) external field. From two distinct routes with different gauge fixings, we derived the SU(2) Kubo formula and showed that those two approaches are equivalent. The non-Abelian feature of SU(2) electric field involves one more term of gauge potentials in comparison to the U(1) case, while this term precisely compensates the nonconservation part in the SU(2) continuity-like equation for the SU(2) current.

For the interests in spin transport, we applied our formula to calculate the spin susceptibility and spin conductivity in the system containing a Rashba or Dresselhaus field. The results show that in the usual system, where $\varepsilon(k) = \hbar^2k^2/2m$, the spin susceptibility is constant. However, the spin conductivity vanishes, much like the case in the spin Hall effect where the spin conductivity in response to the external electric field vanishes in the presence of disorder. To derive the nonvanishing spin conductivity, the systems with nonparabolic unperturbed band structure are necessary, and the spin conductivity, no matter in response to the U(1) or SU(2) electric field, is expected to be observed in such systems. What is more, we also discussed an experimentally available case. In response to the time-dependent Rashba field, the spin conductivity is related to the dielectric function which can be measured directly.

Generalized to the high-dimensional representation, our SU(2) Kubo formula is available to discuss the Luttinger model as well as bilayer spin Hall effect. The spin conductivity in response to the effective field caused by structural inversion asymmetry in the Luttinger model always vanishes.

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APPENDIX A: SPIN CONDUCTIVITY IN SPIN-1/2 REPRESENTATION

The correlation function of spin conductivity in Matsubara formalism reads

$$Q_{ij}^{ab}(i\nu) = -\frac{1}{V}\int_0^\beta du\ e^{i\nu u}\langle T_u \hat{J}_i^{a\dagger}(u)\hat{J}_j^b(0)\rangle, \qquad (A1)$$

where

$$\hat{J}_i^a(u) = \frac{1}{2}\sum_k C_k^\dagger(u)\Big\{\frac{\partial\varepsilon(k)}{\partial k_i} + \frac{\partial d^b}{\partial k_i}\tau^b, \tau^a\Big\}C_k(u). \quad (A2)$$

After using Wick's theorem and introducing the Matsubara function $G(k,u) = -\langle T_u C_k(u)C_k^\dagger(0)\rangle$, one can obtain the correlation function

$$Q_{ij}^{ab}(i\nu) = \frac{1}{4V\beta}\sum_{k,\omega_n} \mathrm{tr}\Big(G\Big\{\frac{\partial\varepsilon(k)}{\partial k_i} + \frac{\partial d^c}{\partial k_i}\tau^c, \tau^a\Big\}G_+\Big\{\frac{\partial\varepsilon(k)}{\partial k_j} + \frac{\partial d^d}{\partial k_j}\tau^d, \tau^b\Big\}\Big), \qquad (A3)$$

where G and G_+ refer to $G(k,i\omega)$ and $G(k,i\omega+i\nu)$, respectively, which can be derived from the Fourier transform

$$G(k,u) = \frac{1}{\beta}\sum_{\omega_n} G(k,i\omega_n)e^{-i\omega_n u}. \qquad (A4)$$

In the case $H_0 = \varepsilon(k) + d^a\tau^a$,

$$\begin{aligned}
G(k,i\omega_n) &= \frac{1}{i\hbar\omega_n + \mu - H_0} \\
&\equiv f(k,i\omega_n)(g(k,i\omega_n) + d^a\tau^a),
\end{aligned} \qquad (A5)$$

with

$$\begin{aligned}
f(k,i\omega_n) &= \frac{1}{(i\hbar\omega_n + \mu - \varepsilon)^2 - |d|^2/4}, \\
g(k,i\omega_n) &= i\hbar\omega_n + \mu - \varepsilon.
\end{aligned} \qquad (A6)$$

In the last line of Eq. (A5), $G(k,i\omega_n)$ is separated into two parts, the U(1) part and SU(2) part, which facilitates our calculation of the trace term,

$$
\begin{aligned}
&\mathrm{tr}\Big[\Big(g(k,i\omega_n)+\frac{1}{2}d^c\tau^c\Big)\Big\{\frac{\partial\varepsilon(k)}{\partial k_i}+\frac{\partial d^c}{\partial k_i}\tau^c,\ \tau^a\Big\}\Big(g(k,i\omega_n+i\nu)+\frac{1}{2}d^c\tau^c\Big)\Big\{\frac{\partial\varepsilon(k)}{\partial k_j}+\frac{\partial d^d}{\partial k_j}\tau^d,\ \tau^b\Big\}\Big]\\
&= 2\Big[\Big(4g(k,i\omega_n)g(k,i\omega_n+i\nu)\delta^{ab}+2d^ad^b-d^2\delta^{ab}\Big)\frac{\partial\varepsilon(k)}{\partial k_i}\frac{\partial\varepsilon(k)}{\partial k_j}+2i\Big(g(k,i\omega_n+i\nu)-g(k,i\omega_n)\Big)\epsilon^{abc}d^c\frac{\partial\varepsilon(k)}{\partial k_i}\frac{\partial\varepsilon(k)}{\partial k_j}\\
&\quad+\Big(g(k,i\omega_n+i\nu)+g(k,i\omega_n)\Big)\Big(\frac{\partial\varepsilon(k)}{\partial k_i}\frac{\partial d^b}{\partial k_j}d^a+\frac{\partial\varepsilon(k)}{\partial k_j}\frac{\partial d^a}{\partial k_i}d^b\Big)+\Big(g(k,i\omega_n)g(k,i\omega_n+i\nu)+|d|^2/4\Big)\frac{\partial d^a}{\partial k_i}\frac{\partial d^b}{\partial k_j}\Big]
\end{aligned}\tag{A7}
$$

Note that summing the Matsubara function over the frequency gives

$$
\begin{aligned}
\frac{1}{\beta}\sum_{\omega_n}f(k,i\omega_n)f(k,i\omega_n+i\nu)g(k,i\omega_n+i\nu) &= -\frac{1}{\beta}\sum_{\omega_n}f(k,i\omega_n)f(k,i\omega_n+i\nu)g(k,i\omega_n)\\
&= \frac{i\hbar\nu(n_{F-}-n_{F+})}{|d|\big[(i\hbar\nu)^2-|d|^2\big]},
\end{aligned}\tag{A8}
$$

and

$$
\begin{aligned}
\frac{1}{\beta}\sum_{\omega_n}f(k,i\omega_n)f(k,i\omega_n+i\nu) &= -\frac{4}{|d|^2\beta}\sum_{\omega_n}f(k,i\omega_n)f(k,i\omega_n+i\nu)g(k,i\omega_n)g(k,i\omega_n+i\nu)\\
&= \frac{2(n_{F-}-n_{F+})}{|d|\big[(i\hbar\nu)^2-|d|^2\big]}.
\end{aligned}\tag{A9}
$$

Thus the last line of the right-hand side of Eq. (A7) vanishes. Changing $i\nu\to\omega$, we can obtain the real and imaginary parts of $Q^{ab}_{ij}(\omega)$,

$$
\begin{aligned}
\mathrm{Re}\,Q^{ab}_{ij}(\omega) &= \frac{1}{2V}\sum_k\frac{\delta^{ab}|d|^2-d^ad^b}{|d|(\hbar^2\omega^2-|d|^2)}\frac{\partial\varepsilon(k)}{\partial k_i}\frac{\partial\varepsilon(k)}{\partial k_j}(n_{F-}-n_{F+}),\\
\mathrm{Im}\,Q^{ab}_{ij}(\omega) &= \frac{\hbar\omega}{2V}\sum_k\frac{\epsilon^{abc}d^c}{|d|(\hbar^2\omega^2-|d|^2)}\frac{\partial\varepsilon(k)}{\partial k_i}\frac{\partial\varepsilon(k)}{\partial k_j}(n_{F-}-n_{F+}).
\end{aligned}\tag{A10}
$$

Since $\sigma^{ab}_{ij}(\omega)=\frac{i}{\hbar\omega}\Big[Q^{ab}_{ij}(\omega)+\frac{\hbar^3n_0}{4m}\delta^{ab}\delta_{ij}\Big]$, then taking the limit $\omega\to0$, we can obtain the dc conductivity in Eq. (32),

$$
\begin{aligned}
\mathrm{Re}\ \sigma^{ab}_{ij} &= -\lim_{\omega\to0}\frac{1}{\hbar\omega}\mathrm{Im}\ Q^{ab}_{ij}(\omega)\\
&= \frac{1}{2V}\sum_k\frac{n_{F-}-n_{F+}}{|d|^3}\epsilon^{abc}d^c\frac{\partial\varepsilon(k)}{\partial k_i}\frac{\partial\varepsilon(k)}{\partial k_j}.
\end{aligned}\tag{A11}
$$

APPENDIX B: SPIN CONDUCTIVITY IN SPIN-3/2 REPRESENTATION

Before we calculate the spin conductivity in spin-3/2 representation, it is wise to warm up with the Clifford algebra. The 4 by 4 gamma matrices Γ^α are constructed by the 2 by 2 sigma matrices, which satisfy $\{\Gamma^\alpha,\Gamma^\beta\}=2\delta_{\alpha\beta}$ and $\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5=-1$. Using these gamma matrices, one can also compose ten antisymmetric matrices $\Gamma^{\alpha\beta}=\frac{1}{2i}[\Gamma^\alpha,\Gamma^\beta]$. Together with the identity matrix, Γ^α and $\Gamma^{\alpha\beta}$ span the space of 4×4 hermitian matrices. The SU(2) generators τ^a in spin-3/2 representations can also be expressed as the linear combinations of $\Gamma^{\alpha\beta}$, i.e., $\tau^a=\frac{1}{4i}L^a_{\alpha\beta}\Gamma^{\alpha\beta}$ with

$$
L^x=\begin{pmatrix}0&0&0&i&i\sqrt{3}\\0&0&-i&0&0\\0&i&0&0&0\\-i&0&0&0&0\\-i\sqrt{3}&0&0&0&0\end{pmatrix},
$$

$$
L^y=\begin{pmatrix}0&0&i&0&0\\0&0&0&i&-i\sqrt{3}\\-i&0&0&0&0\\0&-i&0&0&0\\0&i\sqrt{3}&0&0&0\end{pmatrix},
$$

$$
L^z=\begin{pmatrix}0&-i&0&0&0\\i&0&0&0&0\\0&0&0&-2i&0\\0&0&2i&0&0\\0&0&0&0&0\end{pmatrix}.\tag{B1}
$$

Note that L^a are antisymmetric and satisfy the commutation relation $[L^a,L^b]=i\epsilon^{abc}L^c$. Thus they form the

spin-2 representation of the SU(2) algebra. The following formulas are inevitable in further calculations,

$$\begin{aligned}
&\mathrm{tr}(\Gamma^\alpha\Gamma^\beta)=4\delta^{\alpha\beta}, \qquad \mathrm{tr}(\Gamma^\alpha\Gamma^\beta\Gamma^\gamma)=0,\\
&\mathrm{tr}(\Gamma^\alpha\Gamma^\beta\Gamma^\mu\Gamma^\nu)=4(\delta^{\alpha\beta}\delta^{\mu\nu}-\delta^{\alpha\mu}\delta^{\beta\nu}+\delta^{\alpha\nu}\delta^{\beta\mu}),\\
&\mathrm{tr}(\Gamma^\alpha\Gamma^\beta\Gamma^\gamma\Gamma^\mu\Gamma^\nu)=-4\epsilon^{\alpha\beta\gamma\mu\nu}, \ \mathrm{tr}(\Gamma^{\alpha\beta}\Gamma^\gamma)=0,\\
&\mathrm{tr}(\Gamma^{\alpha\beta}\Gamma^\mu\Gamma^\nu)=4i(\delta^{\alpha\mu}\delta^{\beta\nu}-\delta^{\alpha\nu}\delta^{\beta\mu}),\\
&\mathrm{tr}(\Gamma^{\alpha\beta}\Gamma^\gamma\Gamma^\mu\Gamma^\nu)=4i\epsilon^{\alpha\beta\gamma\mu\nu}.
\end{aligned} \tag{B2}$$

We take the unperturbed and perturbed parts of the Hamiltonian to be $H_0=\sum_k C_k^\dagger\big(\varepsilon(k)+d^\alpha\Gamma^\alpha\big)C_k$ and $H'=\sum_k C_k^\dagger\big(h^\beta\Gamma^\beta\big)C_k$. Accordingly, the Kubo formula for the spin conductivity reads

$$\langle J_i^b(\mathbf{r},t)\rangle=\sigma_i^{b\alpha}(\mathbf{q},\omega)h^\alpha(\mathbf{r},t). \tag{B3}$$

In the limit $q\to 0$,

$$\sigma_i^{b\alpha}(\omega)=\frac{1}{\hbar\omega V}\int_{-\infty}^{t}dt'\ e^{i\omega(t-t')}\langle[\hat{J}_i^b(t),\hat{\Gamma}^\alpha(t')]\rangle_0, \tag{B4}$$

and the corresponding retarded correlation function in Matsubara formalism is given by

$$Q_i^{b\alpha}(\omega)=-\frac{1}{V}\int_0^\beta due^{i\nu u}\langle T_u\hat{J}_i^b(u)\hat{\Gamma}^\alpha(0)\rangle_0. \tag{B5}$$

Similarly, introducing the Matsubara function $G(k,i\omega_n)$ and using the definition of spin current

$$\hat{J}_i^a=\frac{1}{2}\sum_k C_k^\dagger\Big\{\frac{\partial\varepsilon(k)}{\partial k_i}+\frac{\partial d^\beta}{\partial k_i}\Gamma^\beta,\tau^a\Big\}C_k, \tag{B6}$$

we can calculate the trace term

$$\begin{aligned}
&\frac{1}{4i}\mathrm{tr}\Big[\big(g(k,i\omega_n)+d^\mu\tau^\mu\big)\Big\{\frac{\partial\varepsilon(k)}{\partial k_i}+\frac{\partial d^\mu}{\partial k_i}\tau^\mu,\ L^b_{\beta\gamma}\Gamma^{\beta\gamma}\Big\}\big(g(k,i\omega_n+i\nu)+d^\mu\tau^\mu\big)\Gamma^\alpha\Big]\\
=\ &2\Big(g(k,i\omega_n+i\nu)-g(k,i\omega_n)\Big)\Big(2\frac{\partial\varepsilon(k)}{\partial k_i}L^a_{\alpha\beta}d^\beta-\epsilon^{\alpha\beta\gamma\mu\nu}L^a_{\beta\gamma}\frac{\partial d^\mu}{\partial k_i}d^\nu\Big).
\end{aligned} \tag{B7}$$

Note that L^a are all imaginary, then the dc conductivity is given by

$$\sigma_i^{b\alpha}=\frac{-1}{4V}\sum_k\mathrm{Im}\Big(2\frac{\partial\varepsilon}{\partial k_i}L^b_{\alpha\beta}d^\beta-\epsilon^{\alpha\beta\gamma\mu\nu}L^b_{\beta\gamma}\frac{\partial d^\mu}{\partial k_i}d^\nu\Big)\frac{n_{F_-}-n_{F_+}}{|d|^3}. \tag{B8}$$

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